Structural Zeros under the Independence Model

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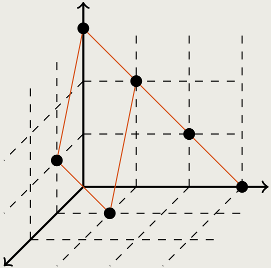
(joint work with Thomas Kahle)

Independence Model - Fibers

Given rows and column sums $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}_0^m$, $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{N}_0^l$, the fiber $\mathcal{F}_{\mathbf{r},\mathbf{c}}$ consists of the following elements:

$$\begin{array}{c|ccccc} & c_1 & c_2 & & c_{l-1} & c_l \\ \hline r_1 & * & * & \ldots & * & * \\ r_2 & * & * & \ldots & * & * \\ & \vdots & \ldots & & \ldots & \vdots \\ r_{m-1} & * & * & \ldots & * & * \\ r_m & * & * & \ldots & * & * \end{array} \in \mathbb{N}_0^{m \times l}$$



Basic moves:

$$\pm \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \ldots$$

Independence Model - Structural Zeros

Given rows and column sums $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}_0^m$,$\mathbf{c} = (c_1, \ldots, c_l) \in \mathbb{N}_0^l$, and $\mathcal{X}_0 \subseteq [m] \times [l]$, the restricted fiber $\mathcal{F}_{\mathbf{r},\mathbf{c}}^{\mathcal{X}_0}$ consists of the following elements:

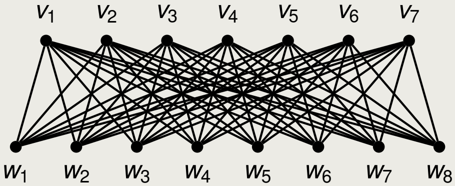
$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & * \\ * & 0 & * & 0 & * & 0 \\ * & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{pmatrix} \end{array} \in \mathbb{N}_0^{7\times 6}$$

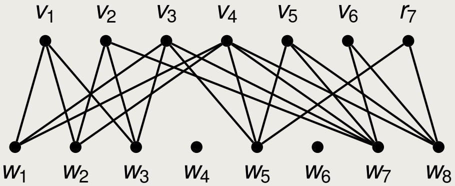
- $\mathcal{F}_{\mathbf{r},\mathbf{c}}^{\mathcal{X}_0} = \mathcal{F}_{\mathbf{r},\mathbf{c}} \cap \{v \in \mathbb{N}_0^{m\times l} : v_{i,j} = 0 \quad \forall (i,j) \in \mathcal{X}_0\}$
- $\ker(A_{m,l}) \cap \{v \in \mathbb{N}_0^{m\times l} : v_{i,j} = 0 \quad \forall (i,j) \in \mathcal{X}_0\} = \ker(A_{m,l}^{\mathcal{X}_0})$

Two-Way Contingency Tables and Bipartite Graphs

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
r_1	?	?	?	?	?	?	?	?
r_2	?	?	?	?	?	?	?	?
r_3	?	?	?	?	?	?	?	?
r_4	?	?	?	?	?	?	?	?
r_5	?	?	?	?	?	?	?	?
r_6	?	?	?	?	?	?	?	?
r_7	?	?	?	?	?	?	?	?

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
r_1	?	?	?	0	0	0	0	0
r_2	0	?	?	0	0	0	?	0
r_3	?	0	?	0	?	0	?	0
r_4	?	?	0	0	?	0	?	?
r_5	0	0	0	0	?	0	?	?
r_6	0	0	0	0	0	0	?	?
r_7	0	0	0	0	?	0	0	?

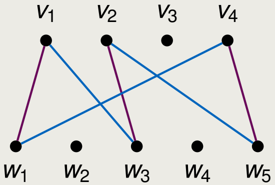




Lets do some (easy) Algebra. . .

$\ldots$ in $K[x_e : e \in E(\mathcal{K}_{m,l})]$

$\mathcal{C} =$



$\mathfrak{p}\left(\mathcal{C}\right) = x_{v_1 w_3} \cdot x_{v_2 w_5} \cdot x_{v_4 w_1} - x_{v_1 w_1} \cdot x_{v_2 w_3} \cdot x_{v_4 w_5}$

$$\mathcal{I}_{A_{m,l}} = \langle \mathfrak{p}\left(\mathcal{C}\right) : \mathcal{C} \text{ cycle of } \mathcal{K}_{m,l} \rangle \in K[x_e : e \in E(\mathcal{K}_{m,l})]$$

$$\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}} = \langle \mathfrak{p}\left(\mathcal{C}\right) : \mathcal{C} \text{ cycle of } \mathcal{K}_{m,l} \setminus \mathcal{X}_0 \rangle \in K[x_e : e \in E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0)]$$

- Rapallo (2006): $\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}} = \mathcal{I}_{A_{m,l}} \cap K[x_e : e \in E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0)]$
- $\mathcal{I}_{A_{m,l}} = \langle \mathfrak{p}\left(\mathcal{C}\right) : \mathcal{C} \text{ 4-cycle in } \mathcal{K}_{m,l} \rangle$
- What about $\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$?

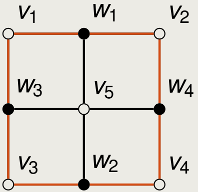
The Markov Basis Property

Question

Under which conditions on $\mathcal{X}_0$ do the 4-cycles of $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$ generate $\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$?

$$\mathcal{I}_k^{\mathcal{X}_0} := \langle \mathfrak{p}(\mathcal{C}) : \mathcal{C} \text{ cycle of } \mathcal{K}_{m,l} \setminus \mathcal{X}_0 \text{ and } |\mathcal{C}| \leq k \rangle$$

$$\mathcal{I}_4^{\mathcal{X}_0} \subseteq \mathcal{I}_6^{\mathcal{X}_0} \subseteq \mathcal{I}_8^{\mathcal{X}_0} \subseteq \cdots \subseteq \mathcal{I}_{2 \cdot \min\{m,l\}}^{\mathcal{X}_0} = \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$$



$$\begin{aligned}\mathcal{I}_4^{\mathcal{X}_0} = \langle & x_{v_1 w_1} x_{v_5 w_3} - x_{v_1 w_3} x_{v_5 w_1}, x_{v_2 w_1} x_{v_5 w_4} - x_{v_2 w_4} x_{v_5 w_1}, \\ & x_{v_5 w_3} x_{v_3 w_2} - x_{v_3 w_3} x_{v_5 w_2}, x_{v_5 w_4} x_{v_4 w_2} - x_{v_4 w_2} x_{v_5 w_2} \rangle \\ = {} & \mathcal{I}_6^{\mathcal{X}_0} \neq \mathcal{I}_8^{\mathcal{X}_0} = \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}\end{aligned}$$

Lemma

$\mathcal{I}_{k-2}^{\mathcal{X}_0} = \mathcal{I}_k^{\mathcal{X}_0}$ if and only if all k-cycles of $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$ have a chord.

The Positive Margins Property

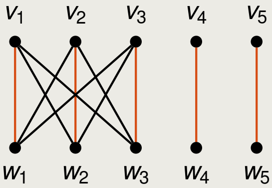
Open Problem (Rapallo, Yoshida; 2009)

Give necessary and sufficient conditions on $\mathcal{X}_0$ so that the remaining basic moves have (PMP).

Theorem (Aoki, Takemura; 2005)

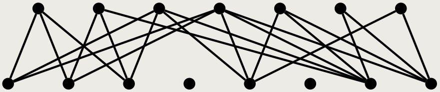
If $\mathcal{X}_0$ only contains diagonal elements, the remaining basic moves have the (PMP).

$$\begin{pmatrix} 0 & ? & ? & ? & ? \\ ? & 0 & ? & ? & ? \\ ? & ? & 0 & ? & ? \\ ? & ? & ? & 0 & ? \\ ? & ? & ? & ? & 0 \end{pmatrix}$$



Good Structures: Coverings

Given $\mathcal{X}_0$.

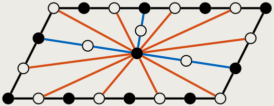


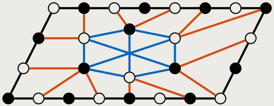
$$\mathcal{I}_4^{\mathcal{X}_0} \subseteq \mathcal{I}_6^{\mathcal{X}_0} \subseteq \mathcal{I}_8^{\mathcal{X}_0} \subseteq \cdots \subseteq \mathcal{I}_{2\cdot\min\{m,l\}}^{\mathcal{X}_0} = \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$$

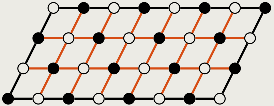
Troublemaker: Chordless Cycles $\mathcal{C}$ with $|\mathcal{C}| > 4$

Theorem

If every chordless cycle $\mathcal{C}$ in $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$ with $|\mathcal{C}| > 4$ has a "good" structure, then the 4-cycles have (PMP).







The Positive Margins Property

Let $\mathcal{I} \subseteq K[x_e : e \in E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0)]$ be an ideal.

$$\mathcal{E}_{\mathcal{I}} := \{e \in E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0) : x_e \notin \mathcal{I}\}$$
$$\hat{\mathcal{E}}_{\mathcal{I}} := \{e \in E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0) : \mathcal{I} = \mathcal{I} : x_e\}$$

Direct consequence from Kahle-Rauh-Sullivant-2012:

Corollary

Let $\mathcal{X}_0$ be such that the 4-cycles generate the cycles space of $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$ and

$$\mathcal{I}_4^{\mathcal{X}_0} = (\cap_{i=1}^r \mathcal{I}_i) \cap \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$$

such that $\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}} \not\subseteq \mathcal{I}_i$ for all $i \in [r]$.

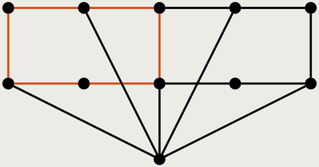
- *$(\mathcal{K}_{m,l} \setminus \mathcal{X}_0) \cap \mathcal{E}_{\mathcal{I}_i}$ has an isolated vertex for all $i \Rightarrow$ (PMP)*
- *(PMP) $\Rightarrow (\mathcal{K}_{m,l} \setminus \mathcal{X}_0) \cap \hat{\mathcal{E}}_{\mathcal{I}_i}$ has an isolated vertex for all i*

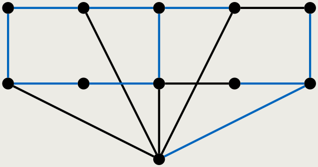
A Simplicial Complex

- Assume, the basic moves generate the integer kernel of $A_{m,l}^{\mathcal{X}_0}$ ($\Leftrightarrow$ the 4-cycles generate the cycles space of $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$)
- Assume, $\mathcal{I}_4^{\mathcal{X}_0}$ is a radical ideal

Let $\mathcal{C}$ be a chordless cycle in $\mathcal{K}_{m,l} \setminus \mathcal{X}_0$ with $|\mathcal{C}| > 4$.

$$\Delta_{\mathcal{C}} := \{\mathcal{M} \subseteq E(\mathcal{K}_{m,l} \setminus \mathcal{X}_0) : \prod_{m \in \mathcal{M}} x_m \cdot \mathfrak{p}(\mathcal{C}) \notin \mathcal{I}_4^{\mathcal{X}_0}\}$$





A Decomposition

Let $\mathcal{E} \in \Delta_{\mathcal{C}}$

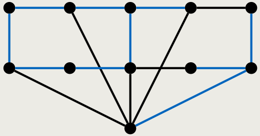
$$\mathcal{I}_{\mathcal{E}} := \bigcap_{\mathcal{M} \subseteq \mathcal{E}} (\mathcal{I}_4^{\mathcal{X}_0} + \langle x_m : m \notin \mathcal{M} \rangle) : (\prod_{m \in \mathcal{M}} x_m)^{\infty}$$

Let $\Delta := \bigcup_{\mathcal{C} \text{ chordless}} \Delta_{\mathcal{C}}$, then

- Eisenbud/Sturmfels (1996): $\mathcal{I}_4^{\mathcal{X}_0} = (\cap_{\mathcal{E} \in F(\Delta)} \mathcal{I}_{\mathcal{E}}) \cap \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$
- Lemma: $\mathcal{I}_{A_{m,l}^{\mathcal{X}_0}} \not\subseteq \mathcal{I}_{\mathcal{E}}$ for $\mathcal{E} \in F(\Delta)$
- Variables not in $\mathcal{I}_{\mathcal{E}}$: $\mathcal{E}$

Corollary

If for all $\mathcal{E} \in F(\Delta)$ there exists an isolated vertex in $(\mathcal{K}_{m,l} \setminus \mathcal{X}_0) \cap \mathcal{E}$, then the 4-cycles have (PMP).

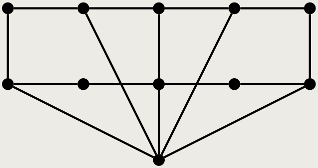


What about Minimal Primes?

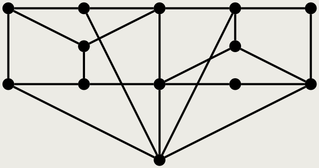
Assume, $\mathcal{I}_4^{\mathcal{X}_0}$ is radical.

$$\mathcal{I}_4^{\mathcal{X}_0} = (\cap_{\mathcal{E} \in F(\Delta)} \mathcal{I}_{\mathcal{E}}) \cap \mathcal{I}_{A_{m,l}^{\mathcal{X}_0}}$$

Do the ideals $\mathcal{I}_{\mathcal{E}}$ for $\mathcal{E} \in F(\Delta)$ coincide with the minimal primes?



YES :)

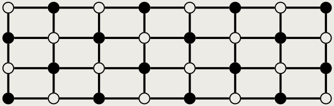


NO :(

But

If Δ is 4-cycle free, the answer is yes!

Grid Graphs



- The 4-cycles generate the cycle space
- The 4-cycles from a Gröbner basis
- $\mathcal{I}_4^{\mathcal{X}_0}$ is a radical ideal

Lemma

The 4-cycles in $4 \times n$ grid graphs have (PMP).

BUT!

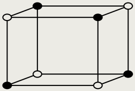
In $5 \times n$ grid graphs, the 4-cycles do not have (PMP).

Radicality

Conjecture

$\mathcal{I}_4^{\mathcal{X}_0}$ is a radical ideal for all $\mathcal{X}_0$.

- Tool: Find a square-free Gröbner basis of $\mathcal{I}_4^{\mathcal{X}_0}$
- Under which conditions will the 4-cycles do the job?



Open Problem

- *Can we determine the minimal primes of $\mathcal{I}_4^{\mathcal{X}_0}$?*
- *When do they coincide with the facets of Δ?*
- *What structures does $\mathcal{X}_0$ have in applications?*