Fiber graphs

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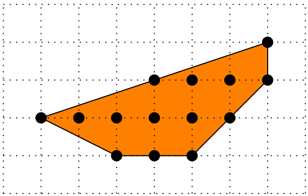
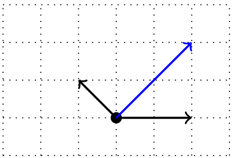
joint work with Caprice Stanley

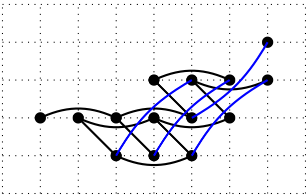
The key player



saturated set, i.e. finite set $\mathcal{F} \subseteq \mathbb{Z}^d$ with $\mathcal{F} = \mathrm{conv}_{\mathbb{Q}}(\mathcal{F}) \cap \mathbb{Z}^d$



set of moves, i.e. finite set $\mathcal{M} \subseteq \mathbb{Z}^d$ without multiples



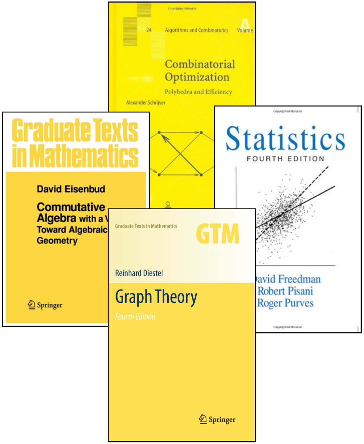
fiber graph $\mathcal{F}(\mathcal{M})$

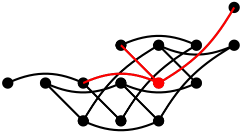
Fiber graph

The *fiber graph* $\mathcal{F}(\mathcal{M})$ is the graph on $\mathcal{F}$ with $u \sim v$ if $u - v \in \pm\mathcal{M}$.

In many applications, nodes are given implicit: $\mathcal{F}_{A,b} := \{u \in \mathbb{N}^d : Au = b\}$.

Applications





- Made for walking (randomly)
- Sampling lattice points
- Log-linear models (goodness-of-fit)
- Solving integer programs

- Statistics: Mixing times?
- Optimization: Number of steps to optimal solution?

Goal

Study the graphs of fiber graphs!

How generic are fiber graphs?

{Fiber graphs}={Simple graphs}!

Observation

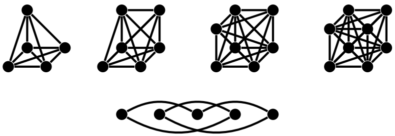
Every simple graph is a fiber graph!

Let $G = (\{v_1, \ldots, v_n\}, E)$ be a graph and define $\mathcal{F} := \{e_1, \ldots, e_n\} \subseteq \mathbb{Z}^n$ and $\mathcal{M} := \{e_i - e_j : v_i v_j \in E\}$, then $G \cong \mathcal{F}(\mathcal{M})$.

Fiber dimension of a graph

... is the smallest $d \in \mathbb{N}$ such that G is isomorphic to a fiber graph in $\mathbb{Z}^d$.

- $\mathrm{fdim}(K_n) = \lceil \log_2(n) \rceil$
- $\mathrm{fdim}(C_n) = 1 \Leftrightarrow n \notin \{3, 4, 6\}$



Advertisement: W., “The fiber dimension of a graph” (arXiv:1601.04527)

Connectedness

Markov basis

A set $\mathcal{M} \subseteq \mathbb{Z}^d$ is a *Markov basis* for a collection $\mathfrak{F}$ of saturated sets in $\mathbb{Z}^d$ if $\mathcal{F}(\mathcal{M})$ is connected for all $\mathcal{F} \in \mathfrak{F}$.

Applications: $\mathfrak{F}_A := \{\mathcal{F}_{A,b} : b \in \mathbb{N}A\}$.

Fundamental theorem (Diaconis, Sturmfels; 1998)

A set $\mathcal{M} \subseteq \mathbb{Z}^d$ is a Markov basis for $\mathfrak{F}_A$ if and only if $\mathcal{I}_{\mathcal{M}} = \mathcal{I}_A$.

- We can do better (Rauh, Sullivant; 2015):

$$\mathfrak{F}_{\mathcal{L},D} := \{\{(u + \mathcal{L}) \cap \{v \in \mathbb{Z}^d : Dv \leq c\} : u \in \mathbb{Z}^d, c \in \mathbb{Z}^m\}$$

- $\mathfrak{F}_{\ker_{\mathbb{Z}}(A),-I} = \mathfrak{F}_A$

Universality theorem (De Loera, Onn; 2006)

Markov bases of $(3, r, c)$-tables with fixed 2-margins are arbitrarily complicated.

Diameter

$$\mathrm{diam}\,(\mathcal{F}(\mathcal{M})) \geq \frac{\max\{\|u-v\|_1 : u,v \in \mathcal{F}\}}{\max\{\|m\|_1 : m \in \mathcal{M}\}}$$

Let $\mathfrak{F}$ be a collection of saturated sets in $\mathbb{Z}^d$.

Norm-reducing

A set $\mathcal{M}$ is *norm-reducing* for $\mathfrak{F}$ if for all $\mathcal{F} \in \mathfrak{F}$ and $u, v \in \mathcal{F}$, there is $m \in \mathcal{M}$ such that $u + m \in \mathcal{F}$ and $\|u + m - v\|_1 < \|u - v\|_1$.

- $\mathrm{diam}\,(\mathcal{F}(\mathcal{M})) \leq \max\{\|u-v\|_1 : u,v \in \mathcal{F}\}$ for norm-reducing $\mathcal{M}$
- Graver basis is norm-reducing for $\mathfrak{F}_A$.
- $\{$Norm-reducing for $\mathfrak{F}\} \subseteq$? $\subseteq \{$Markov basis for $\mathfrak{F}\}$

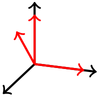
Norm-like

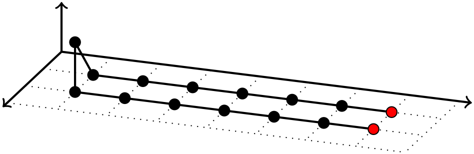
A set $\mathcal{M}$ is *norm-like* for $\mathfrak{F}$ if there is a constant $C \in \mathbb{Q}$ such that for all $\mathcal{F} \in \mathfrak{F}$ and all $u, v \in \mathcal{F}$, $\mathrm{dist}_{\mathcal{F}(\mathcal{M})}(u, v) \leq C \cdot \|u - v\|_1$.

- $\mathrm{diam}\,(\mathcal{F}) \sim \max\{\|u-v\|_1 : u,v \in \mathcal{F}\}$
- $\{$Norm-reducing for $\mathfrak{F}\} \subseteq \{$Norm-like for $\mathfrak{F}\} \subseteq \{$Markov basis for $\mathfrak{F}\}$

Diameter

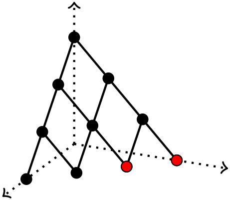
- Markov basis $\not\Rightarrow$ Norm-like





- Norm-like $\not\Rightarrow$ Norm-reducing

$$\mathcal{F}_i := \{u \in \mathbb{N}^3 : u_1 + u_2 + u_3 = i\}$$
$$\mathcal{M} := \{e_3 - e_2, e_3 - e_1\}$$



Theorem (Stanley, W.; 2016)

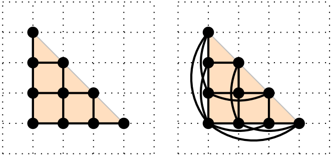
When $\mathfrak{F} = \mathfrak{F}_{A_1} \cup \cdots \cup \mathfrak{F}_{A_r}$, then every Markov basis for $\mathfrak{F}$ is norm-like.

- $C' \cdot i \leq \operatorname{diam}(\mathcal{F}_{A,ib}(\mathcal{M})) \leq C \cdot i$
- Fiber walks in fixed dimension mix slowly

Improving the diameter

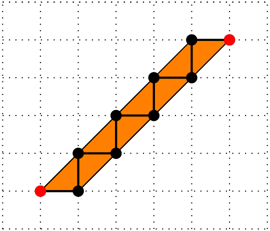
Compressed fiber graphs

The *compressed fiber graph* $\mathcal{F}^c(\mathcal{M})$ is the graph on $\mathcal{F}$ with $u \sim v$ if $u - v \in \{\lambda \cdot m : m \in \mathcal{M}, \lambda \in \mathbb{Z}\}$.



Theorem (Sebö; 1990)

For any $A \in \mathbb{Z}^{m \times d}$ and $\mathcal{F} \in \mathfrak{F}_A$,
$\operatorname{diam}(\mathcal{F}^c(\mathrm{Gr}_A)) \leq 2d - 2$.



Theorem (Stanley, W.; 2016)

For any Markov basis $\mathcal{M}$ of $\mathfrak{F} = \mathfrak{F}_{A_1} \cup \cdots \cup \mathfrak{F}_{A_r}$, there is a constant C such that for all $\mathcal{F} \in \mathfrak{F}$,

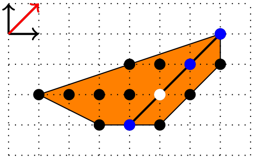
$$\operatorname{diam}(\mathcal{F}^c(\mathcal{M})) \leq C.$$

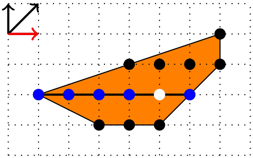
Walking on compressed fiber graphs

Heat-bath walk

Let $\mathcal{F}, \mathcal{M} \subseteq \mathbb{Z}^d$ be finite and let $\pi : \mathcal{F} \to [0,1]$ and $f : \mathcal{M} \to [0,1]$ be mass functions, then

$$\mathcal{H}(u,v) = \begin{cases} f(m) \cdot \frac{\pi(v)}{\pi((u+m\cdot\mathbb{Z})\cap\mathcal{F})} & \text{, if } v \in u + \mathbb{Z}\cdot m \\ 0 & \text{, otherwise.} \end{cases}$$

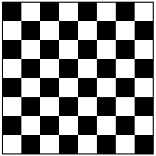




Basics

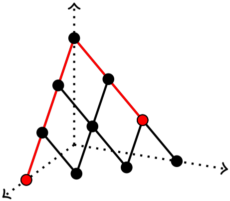
- Many names: Glauber dynamics, Hit-and-run, ...
- $\mathcal{H}$ is irreducible, reversible, and converges to π, $\mathrm{Spec}(\mathcal{H}) \subseteq [0,1]$
- Rook's walk is a special Heat-bath walk

of



Mixing time of Rook's walk

Heat-bath walk on $[n]^d$: $\text{SLEM}(\mathcal{H}) = 1 - \frac{1}{d}$

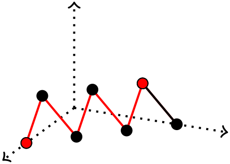


Augmenting Markov bases

A Markov basis $\mathcal{M}$ is *augmenting* for $\mathcal{F}$ if for all $u, v \in \mathcal{F}$, there is a path in $\mathcal{F}^c(\mathcal{M})$ of the form

$$u \to u + \lambda_1 m_1 \to u + \lambda_1 m_1 + \lambda_2 m_2 \to \cdots \to v$$

with **distinct** moves from $\mathcal{M}$. Let $\mathcal{L}_{\mathcal{M}}(\mathcal{F})$ be the maximum length of all minimal augmenting paths.



- In general "$\mathcal{L}_{\mathcal{M}}(\mathcal{F}) \geq \dim(\mathcal{F})$"

Heat-bath walk in fixed dimension

Theorem (Stanley, W.; 2016)

Let $\mathcal{M}$ be augmenting for $\mathcal{F} \subseteq \mathbb{Z}^d$, $r_i := \max\{|(u + m_i \cdot \mathbb{Z}) \cap \mathcal{F}| : u \in \mathcal{F}\}$, and suppose that $r_1 \geq r_2 \geq \cdots \geq r_k$. Then

$$\mathrm{SLEM}(\mathcal{H}) \leq 1 - \frac{|\mathcal{F}| \cdot \min(f)}{C_{\mathcal{L}_{\mathcal{M}}(\mathcal{F})} \cdot r_1 r_2 \cdots r_{\mathcal{L}_{\mathcal{M}}(\mathcal{F})}}.$$

Corollary

Let $\mathcal{P} \subseteq \mathbb{Q}^d$ be a polytope and $\mathfrak{F} := \{(i \cdot \mathcal{P}) \cap \mathbb{Z}^d : i \in \mathbb{N}\}$. Let $\mathcal{M}$ be augmenting for $\mathfrak{F}$ with $\mathcal{L}_{\mathcal{M}}(\mathcal{F}) \leq \dim(\mathcal{P})$ for all $\mathcal{F} \in \mathfrak{F}$, then the Heat-bath walk on $\mathcal{F}^c(\mathcal{M})$ is a spectral expander (faster than rapidly mixing).

- Many $0/1$ Graver bases have augmentation length $\leq \dim \ker_{\mathbb{Z}}(A)$
- Markov basis for $A = (1, \ldots, 1)$
- Unit vectors for cross-polytope, truncated hyperrectangeles, ...

Wrap up

Take home messages

- We need a better graph-understanding of fiber graphs
- Every simple graph is a fiber graph
- Compressed fiber graphs are cool. . . and they better take a heat-bath!

Open problems

- Conductance/Edge-expansion of compressed fiber graphs?
- Connectivity of compressed fiber graphs?
- Can any Markov basis made augmenting by adding finitely many moves?